An M/M(μ_1)+M(μ_2)/1 Queue with a State Dependent N-Policy

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Abstract

Here in, we derive conditional distributions of waiting times for customers of a single server Poisson queue with a state dependent N-policy. If the number Q of customers present in the system belongs to the subspace Ω_1={0,1, ...,N}, the mean service rate is μ_1 and if the same Q moves to the subspace Ω_2={(N+1), (N+2), ..., ∞}, the service rate changes to μ_2 for some $\mu_2 = (\mu_1 + \mu) \geq \mu_1$. In each case, service time distribution follows an exponential distribution. The inter-arrival times follow an exponential distribution with parameter λ for all Q $\in \Omega_1 + \Omega_2$. Thus the proposed queue ia modeled as M/M(μ_1)+M(μ_2)/1/N-policy. For various combinations of N, $\rho_1 = \frac{\lambda}{\mu_1} \neq 1$ and $\rho = \frac{\lambda}{\mu_2} < 1$, we derive steady state queue length and waiting time distributions. Further, we compute conditional mean queue lengths, and mean waiting times for a given value of N to establish the Little's formula. We also identify a linking factor U(N) that enables us to derive the waiting time of the proposed system as a function of the waiting times of M/M(μ_1)/1 and M/M(μ_2)/1 models. A unique turning integer point N* is then found such that U(N*)$\geq$ U(N) for N $\in \Omega_1 \cup \Omega_2$, *i.e.* $\max_N$ U(N)=U(N*). For a real life aplication, by fixing the revenue per unit time wait, a maximization problem is formulated in terms of U(N). An optimal value N* for N is determined by a graphical illustration that maximizes the total expected revenue (TER) per unit time wait.
Keywords: Fast and Slow Service Rates; Conditional Mean System Size, Conditional Mean Waiting Time, Little's Formula.
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1 Introduction

Queueing models with fast and slow servers have been investigated since 1960s. Gumbel (1960) has discussed a 'multiple-booth' queue served by non-homogeneous servers. Krishnamoorthi (1963) has developed a procedure to modify the classical FCFS (First-Come-fist-served) rule for Poisson queues with an N-policy for the non-violation of the Little's

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formula. For M/Mi/2/N-policy and M/Mi/3/N-policy queues, Singh (1970, 1971) has used easily implementable numerical algorithms. Lin and Kumar (1984) have established that the optimal N-policy for controling of a single queue served by two heterogeneous servers is of threshold type. Rubinovitch (1985a 1985b) has identified ideal conditions to install a slow server into a queueing system that has initially one fast server only. Shawky (1999) has studied a slow server queue to obtain a precise solution. Cabral (2005) has discussed multi-server queues for uninformed customers served by slow servers. Abou-El-Ata and Kim et al. (2011) have analysed M/M/m queues with servers of different speeds. Zhang and Daigle (2012) have suggested an optimal method that minimizes the long-run average number of jobs served by multiple number of heterogeneous servers.

Using matrix analytical methods of Neuts(1981), Yue et al. (2015) have investigated a 2-server queuing system with an N-policy and have proposed a cost function to optimize the effect of the parameters. Sivasamy et al. (2015, 2016, 2020) also have investigated queues with fast and slow servers and with impatient customes.

Zhang and Wang (2015) have explored various properties of a Poisson queue M/M/1 operating under a threshold type of T-policy together with some useful applications. The primary objective of this article is to investigate the same type of M/M/1 queue to derive some more new results and closed-form expressions for many metrics like the mean waiting times. We study the continuous time queue length (queue+service) process $\{$Q(t)$\}$ on the state space $\Omega_1 \cup \Omega_2$= $\{$0,1,2,. . .,N,(N+1),(N+2), $\ldots, \infty\}$. Thus the set of all communicating states is on the space $\Omega_1 \cup \Omega_2$. When the sequence $\{$Q(t)$\}$ constitues an ergodic process, there is a pre-arrival epoch to the state (N+1) with positive probability of occurence which we denote by (N+1)$^{(-)}\}$

One motivating factor is to check conditional Littl's formula in the sequal. If the system is in equilibrium enviroment *i.e.* when Q= $\lim_{t\to\infty}$ Q(t) $\in \Omega_1 \cup$ (N+1)$^{(-)}\}$, the service rate remains at a lower value μ_1 and if Q $\in \{$(N+1),(N+2,. . . , $\infty\}$, the service rate is changed to a new value $\mu_2 = (\mu_1 + \mu)$ for some $\mu \geq 0$. It is remarked that the service rate for an arriving customer is μ_1 just before joining the state (N+1) but the service rate for the same customer is μ_2 just after joining the state (N+1). Queue length distribution is determined by balancing the steady state system of diffrence equations and solving them by the classical iterative method while the (conditional) waiting time distributions by the Laplace Transform methods.

Section 2 describes the basic details of M/M(μ_1)+M(μ_2)/1 queue with an N-Policy. Section 3 presents the compact statements on the conditional and uncondtional mean system size (queue+service) and mean sojurn times. Little's formula is then verified for a given N-policy. Section 4 highlights an application, where total expected revenue (TER)

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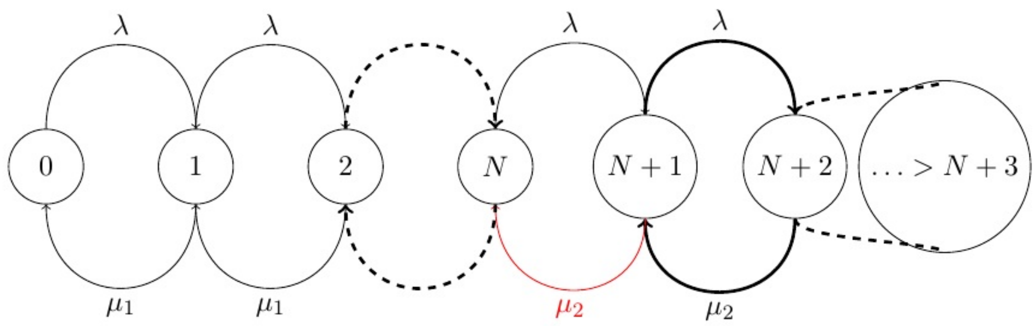
Figure 1: State Space diagram for M/M(μ_1)+M(μ_2)/1/ N-policy queue



is maximized over a graphical chart. Section 5 provides a formal concluding results and scope.

2 Queue Length Distribution of M/M(μ_1)+M(μ_2)/1/N-policy Queue

The transitions between succesive states of the queue length process Q $\in \Omega \cup \Omega$ of the queue M/M(μ_1)+M(μ_2)/1/N-policy under study is displayed in Figure1. The N-policy operates subject to :

- Customers arrive to an M/M(μ_1)+M(μ_2)/1/N-policy facility at rate λ according to a Poisson process
- Queue discipline is the classical first-come first-served (FCFS).
- The queue length process { Q(t) } has a stumbling state (N+1)$^{(-)}$ which can move on to the state N at rate μ_1 but not to the state (N+1).
- Service times are exponentially distributed with a parameter $\mu_1 > 0$ (slow rate) for all customers of Q $\in \Omega_1$ and with a parameter μ_2= $(\mu + \mu_1)$ (fast rate), where $\mu > 0$, for all customers of Q $\in \Omega_2$.

Let $\mu_2 > \mu_1$, $\mu_1 > 0$ and $\mu > 0$ and

$$\rho_1 = \frac{\lambda}{\mu_1} \neq 1 \;\; and \;\; \rho = \frac{\lambda}{\mu_2} < 1 \tag{2.1}$$

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Theorem 2.1. *Let π_n denote the probability of n customers in the system subject to the constraint ρ <1. The balance system of equations describing the above M/M(μ_1)+M(μ_2)/1 queueing system under the N-policy are given by*

$$\lambda\ \pi_0 = \mu_1\ \pi_1 \tag{2.2}$$

$$(\lambda + \mu_1)\ \pi_n = \lambda\ \pi_{(n-1)} + \mu_1\ \pi_{(n+1)}\ \ for\ n = 1, 2, \ldots, (N-1) \tag{2.3}$$

$$(\lambda + \mu_1)\ \pi_N = \lambda\ \pi_{(N-1)} + \mu_2\ \pi_{(N+1)} \quad for\ n = N \tag{2.4}$$

$$(\lambda + \mu_2)\ \pi_n = \lambda\ \pi_{(n-1)} + \mu_2\ \pi_{(n+1)}\ \ for\ n = (N+1), (N+2), \ldots, \infty \tag{2.5}$$

Proof. By usual probabilistic arguments, we get the system of difference equations in (2.2) to (2.5). □

Theorem 2.2. *Let the normalizing condition be*

$$\sum_{0}^{N-1} \pi_n + \sum_{N}^{\infty} \pi_n = 1 \tag{2.6}$$

The unique solution set $\{\pi_n\}$ that satisfies the system of equations (2.2) *through* (2.5) *is obtained as given below:*

$$\pi_i = \pi_0\ \rho_1^i \quad for\ i \in \{0, 1, 2, \ldots, N \tag{2.7}$$

$$\pi_i = \pi_N\ \rho^{(i-N)}\ \ for\ i \in \{(N+1), (N+2), \ldots, \infty\}. \tag{2.8}$$

and

$$\pi_0 = \frac{(1-\rho)(1-\rho_1)}{(1-\rho_1^N)((1-\rho) + (1-\rho_1)\rho_1^N)} \tag{2.9}$$

Proof. From (2.2), we get

$$\pi_1 = \pi_0 \frac{\lambda}{\mu_1} = \pi_0\ \rho_1$$

Putting n = 2 in (2.3) then solving it using $\pi_1 = \pi_0\ \rho_1$ we get

$$\pi_2 = \pi_0\ \rho_1^2$$

Continuing this induction process by assigning n=3, 4, . . .,N in (2.3), (2.4) and solving them iteratively we can establish (2.7) and (2.8).
Putting n = N+1 in (2.5) and solving it using $\pi_N = \pi_0 \rho_1^N$ we get

$$\pi_{N+1} = \pi_0\ \rho_1^N \rho$$

Repeating this induction process iteratively with n=(N+2), (N+3), . . . in (2.5), and solving them we can establish (2.8) and then (2.9) on using (2.6). □

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Corollary 2.3. *The conditional probability $B_{(<N)}$ that an arriving customer will find j $\in \{0, 1, 2, \ldots, (N-1)\}$ number of customers in the waiting line is given by*

$$B_{(<N)} = \sum_{0}^{(N-1)} \pi_n = \pi_0 \, \frac{1-\rho_1^N}{1-\rho_1} \tag{2.10}$$

Corollary 2.4. *The conditional probability $B_{(\geq N)}$ that an arriving customer will find j number of customers in the subspace $\{$ N,(N+1),$\ldots$,$\infty\}$ is given by*

$$B_{(\geq N)} = \sum_{N}^{\infty} \pi_n = \pi_0 \, \frac{\rho_1^N}{1-\rho} \tag{2.11}$$

and $B_{(<N)}+B_{(\geq N)}=1$ which in turn gives the value of π_0 as stated in (2.9).

Particular cases:

1. When N=∞, then $B_{(\geq N=\infty)}$=0 and π_0=(1-ρ_1). All customers are served by the classical M/M(μ_1)/1 service facility. Thus the mean system size $L_s^{(1)}$ is

$$L_s^{(1)} = \frac{\rho_1}{1-\rho_1} \tag{2.12}$$

2. When N=0, then $B_{(<N=0)}$=0 and π_0=(1-ρ). All customers are served by the classical M/M(μ_2)/1 system. Thus the mean system size $L_s^{(2)}$ is

$$L_s^{(2)} = \frac{\rho}{1-\rho} \tag{2.13}$$

3 The Conditional Little's Law for a given N-policy

A special faeture is that the probability of an arriving customer to see the system M/M(μ_1)/1, just before joining the sate (N+1) equals $\pi_0 \ \rho_1^{(N+1)}$ but just after joining the state (N+1), the same probability is changed to $\pi_0 \ \rho_1^N \ \rho$.

Theorem 3.1 (**The Conditional Little's law).** *The conditional expected number of customers to be observed by a random arrival is*

$$L = (\frac{\rho_1}{1-\rho_1})\{B_{(<N)} - N\pi_0\rho_1^N\} + B_{(\geq N)}\{N\rho_1 + \frac{\rho}{1-\rho}\} \tag{3.1}$$

Proof. The conditional expected number L_1 of customers to be observed by an arriving customer who would be served at the N^{th} departure epoch is $L_1=\sum_0^N n\pi_n$.

$$L_1 = \pi_0 \left(\frac{\rho_1[1-\rho_1^N - N(1-\rho_1)\rho_1^N]}{(1-\rho_1)^2} \right) = \left(\frac{\rho_1}{1-\rho_1} \right) \left(B_{(<N)} - N\pi_0\rho_1^N \right) \tag{3.2}$$

Further, the conditional expected system size L_2 to be observed by an arriving customer when the system state is between N and (N+1) with probability $\pi_N\rho_1 = \pi_0\rho_1^{N+1}$ is

$$L_2 = N\pi_0\rho_1^{(N+1)} + \pi_0\rho_1^N \sum_{n=N+1}^{\infty} n\rho^{(n-N)}$$

$$= N\pi_0\rho_1^N \frac{\rho_1}{(1-\rho)} + \pi_0\rho_1^N \frac{\rho}{(1-\rho)^2} = B_{(\geq N)} \left(N\rho_1 + \frac{\rho}{(1-\rho)} \right) \tag{3.3}$$

Thus $L=L_1+L_2$ is given by

$$L = (\frac{\rho_1}{1-\rho_1})\{B_{(<N)} - N\pi_0\rho_1^N\} + B_{(\geq N)}\{N\rho_1 + \frac{\rho}{1-\rho}\} \tag{3.4}$$

□

Remarks:

1. For the range of $1 \leq N < \infty$, the mean system size of (3.4) can be reorganised as in (3.5):

$$L = B_{(<N)}\frac{\rho_1}{(1-\rho_1)} + B_{(\geq N)}\frac{\rho}{(1-\rho)} + N\pi_0\rho_1^N \left((\frac{\rho_1}{1-\rho}) - (\frac{\rho_1}{1-\rho_1}) \right) \tag{3.5}$$

2. When $\mu_1 > 0$, and $\mu = 0$ then $\rho_1=\rho$. Thus the mean system size of M/M/1 model obtained from (3.5) matches with (2.12).

3.1 Conditional and Unconditional Waiting Time Distributions

The Little's formula states that the long-term average number L of customers in a stationary system is equal to the long-term average effective arrival rate λ multiplied by the average time $\bar{W}$ that a customer spends in the system *i.e.* L = $\lambda\ \bar{W}$. This section explains how an arriving customer justifies this formula upon arrival into the proposed M/M(μ_1)+M(μ_2)/1 system with N-policy (0< N $\leq \infty$)

Let $H_k(G_k)$ be the sum of k independent and indentically distributed (*iid*) random variables having a common exponential distribution with mean $1/\mu_1$ ($1/\mu_2$). Thus H_k has an

Erlang-k distribution with mean (k/μ_1) and G_k has an *Erlang-k* distribution with mean (k/(μ_2)) for k=1,2,. . ..

PASTA (Poisson Arrivals See Time avrerages): Consider a typical customer entering the system in steady state. The PASTA ensures that the random queue length, L, seen by a typical customer upon arrival (in the case of Poisson arrivals only) is the same as the steady state queue length. Hence, if we see that the mean response time is $\bar{W}$ then L= $\lambda\,\bar{W}$, where λ being the effective arrival rate.

Let the conditional cumulative distribution of the waiting(or sojourn) time for an arriving cutomer who finds (N-1) customers ahead just before joining the queue be denoted by W(t). For this given condition, the PASTA property leads to the queue length as a measure of performance seen from the viewpoint of the system operator.

$$\frac{d}{dt}W(t) = \sum_{n=0}^{(N-1)} \pi_n \; P(\sum_{k=1}^{(n+1)} H_k = t) \tag{3.6}$$

$$\Longrightarrow \; w(t) = \sum_{n=0}^{(N-1)} \pi_0 \; \rho_1^n \sum_{k=1}^{(n+1)} \mu_1 \; \frac{(\mu_1 t)^{(k-1)}}{(k-1)!} e^{-(\mu_1\, t)}. \tag{3.7}$$

The Laplace Transform of (3.7) is given by w*(s) *i.e.*

$$w^*(s) = \pi_0 \; (\frac{\mu_1}{\mu_1 + s}) \; \frac{1 - (\frac{\mu_1\,\rho_1}{\mu_1+s})^N}{1 - (\frac{\mu_1\,\rho_1}{\mu_1+s})}. \tag{3.8}$$

Thus the conditional mean time $\bar{W}^{(1)}$ to be spent in the system is $\bar{W}^{(1)}$= - $\frac{d}{ds}$ w*(s)$_{s=0}$ $\Longrightarrow$

$$\bar{W}^{(1)} = \frac{1}{\mu_1\;(1-\rho_1)} \left(B_{(<N)} - N\pi_0\rho_1^N\right) = (\frac{1}{\lambda})\; L_1. \tag{3.9}$$

Multiplying the conditional the average time $\bar{W}^{(1)}$ that a customer spends in the system given in (3.9) by the mean arrival rate λ we see that the result is L_1 = $\lambda\,\bar{W}^{(1)}$. This is an example for conditional Little's law.

Justification: When $\rho = \frac{\lambda}{\mu_2} < 1$, denote the conditional cumulative distribution of the waiting (or sojourn) time for an arriving cutomer who finds the system state in the subspace $\{$N,(N+1),. . . ,∞ $\}$ by $W^{(2)}$(t).

$$\frac{d}{dt}W^{(2)}(t) = \sum_{n=N}^{\infty} \pi_n \; P(H_N + \; G_{(n+1-N)} = t) \tag{3.10}$$

The Laplace Transform of (3.10) is given by $w^{*(2)}$(s) *i.e.*

$$w^{*(2)}(s) = \sum_{n=N}^{\infty} \pi_0\rho_1^N \rho^{(n-N)}(\frac{\mu_1}{\mu_1+s})^N \; (\frac{\mu_2}{\mu_2+s})^{(n+1-N)}. \tag{3.11}$$

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i.e.

$$w^{*(2)}(s) = \pi_0 \; (\frac{\mu_1 \; \rho_1}{\mu_1 + s})^N \; (\frac{\mu_2}{\mu_2(1-\rho)+s}). \tag{3.12}$$

The conditional mean time $\bar{W}^{(2)}$ to be spent in the system for an arriving customer C_N joins the N^{th} position of the queue is obtained by setting s=0 on the negative first derivative of $w^{*(2)}(s)$ *i.e.* $\bar{W}^{(2)}$ = -$\frac{d}{ds}$ $w^{*(2)}(s)$ at $_{s=0}$ which $\Longrightarrow$

$$\bar{W}^{(2)} = B_{(\geq N)} \left(N\frac{1}{\mu_1} + \frac{1}{\mu_2(1-\rho)} \right) = \frac{1}{\lambda} \; L_2. \tag{3.13}$$

Hence, comparing the mean system size L_2 of (3.3) and mean time spent $\bar{W}^{(2)}$ for the given conditions of (3.13), we conclude that the conditional Little's law is justfied.

The condinal mean sojourn time $\bar{W}^{(2)}$ of (3.13) corresponds to the mean waiting time of a customer who finds the system state at N just after her arrival epoch and is served as the (N+1)st bcustomer by served by M/M(μ_1)/1 system with probability $B_{(\geq N)}$. Using the mean value approach, we estimate the expected time spent by the front line customers of size N is N($\frac{1}{\mu_1}$). Using the Lithettle's law, the second term of (3.13) corresponds to the mean time spent by an arbitrary cutomer served by M/M(μ_2)/1 system with probability $B_{(\geq N)}$.

3.2 Properties of U_1(N)=Nπ_0 and U(N)=U_1(N) ρ_1^N, for N$\geq$1, ρ <1 and $\rho_1 \neq$1

Let us consider a linking factor U(N) as follows:

$$U_1(N) = N\pi_0, \tag{3.14}$$
$$U(N) = U_1(N)\rho_1^N = N\pi_0\rho_1^N. \tag{3.15}$$

Thus the unconditional mean waiting time $\bar{W} = \bar{W}^{1)} + \bar{W}^{(2)}$ is given by

$$\bar{W} = B_{(\geq N)} \left(\frac{1}{\mu_2(1-\rho)} \right) + B_{(<N)} \left(\frac{1}{\mu_1 \; (1-\rho_1)} \right)$$
$$+U(N)\{\frac{1}{\mu_1 \; (1-\rho)} - \frac{1}{\mu_1 \; (1-\rho_1)}\} = \frac{L}{\lambda}. \tag{3.16}$$

A rule of thumb is the Poisson distribution is a decent approximation of the Binomial if n $\geq$ 20 and np $\leq$ 10. Suppose a care taker of the proposed M/M(μ_1)+M(μ_2)/1 system with N-policy conducts series of 'Bernouli trials' at random instances to check if the system is in Idle (success) state or otherwise. Thus the success probability is π_0= P(system is in Idle state) and that of failure event is (1-π_0). If this trial is repeated N number of times independently then the expected value is given by U_1(N) of (3.14).

Table 1: Optimal policy N* giving maximum of Nπ_0 values for λ=5.0, μ=1.1, $\mu_1 \in \{4,4.2,4.51,4.8 \text{ and } 4.9\}$

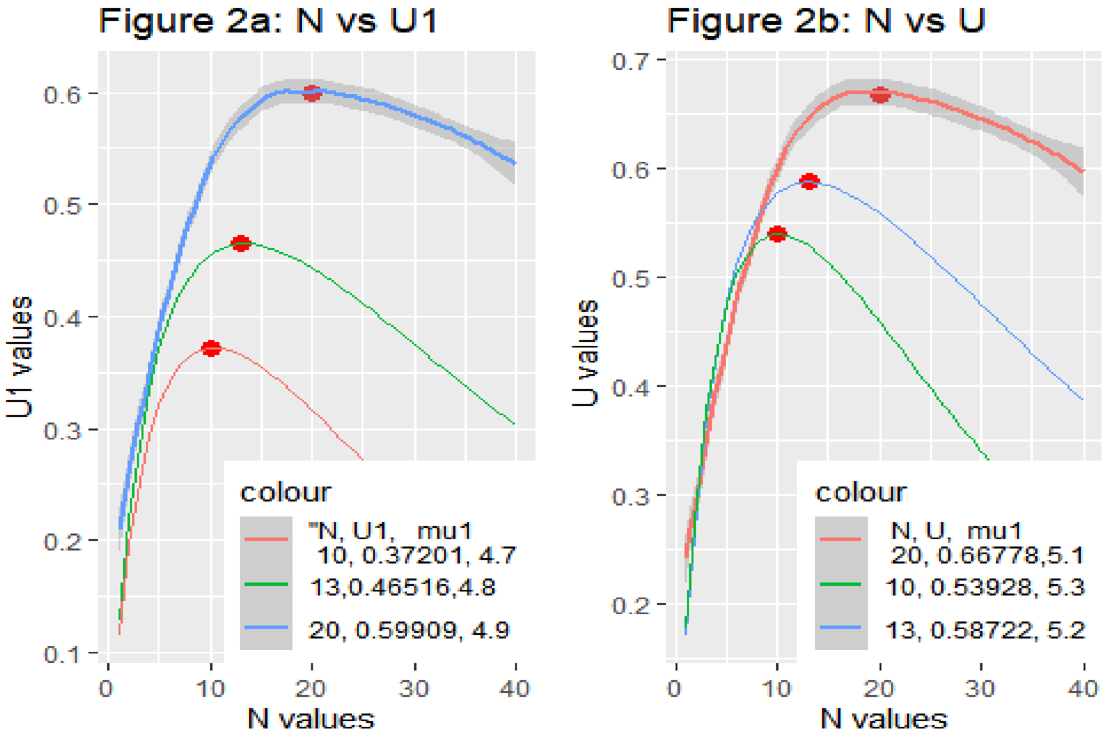
N	μ_1=4	μ_1=4.2	μ_1=4.51	μ_1=4.7	μ_1=4.8	μ_1=4.9
2	0.024409	0.073453	0.149117	0.194763	0.218462	0.241902
4	0.030704*	0.098089	0.215126	0.29228	0.334019	0.376325
5	0.030516	0.100915*	0.231333	0.321357	0.371076	0.422082
6	0.029153	0.100026	0.240364	0.341838	0.399048	0.458442
7	0.027104	0.096672	0.244121	0.355826	0.420111	0.487639
8	0.024705	0.091741	0.243979*	0.364827	0.435812	0.511252
9	0.022179	0.085868	0.24095	0.369946	0.447285	0.530435
10	0.019676	0.079506	0.235791	0.372014*	0.455378	0.546046
11	0.017288	0.072976	0.22908	0.371665	0.460738	0.569017
12	0.015069	0.066502	0.221258	0.369394	0.463866	0.473913
13	0.013046	0.060237	0.212669	0.365587	0.465159*	0.577281
14	0.011231	0.05428	0.20358	0.360556	0.464932	0.583845
15	0.00962	0.048693	0.194198	0.354548	0.463441	0.588967

As the values of π_0 decrease with increasing values of N from N=1, it can be shown that U_1(N) is a concave function for each integer value of N . The value N* of N that corresponds to the maximum of U_1(N)=0.030704 for $\rho_1 > 1$ is N*=3 when λ=5.0, μ_1=4, μ_1=1.1 which is shown in Table1. The system M/M(μ_1)+M(μ_2)/1 under study moves to the Idle state more number of times (which in turn minimizes the waiting time of customers) as compared with values of N$\neq$3.
The maximum value of U_1(N) for λ=5.0, $\mu_1 \in \{4.2,4.51,4.7,4.8,\text{and } 4.9\}$, μ_1=1.1 is marked with a * in the Table1 for which value N* of N is 5,8,10,13, and "nil" respectively.
The maximum value of U_1(N) for λ=5.0, $\mu_1 \in \{4.9,4.8,4.7\}$, μ_1=1.1 is represented in the Figure2a where N varies from 1 to 40. From the Figure2b we observe the following pairs (N, U_1(N)) that

(i) When μ_1=4.9, the maximum of U_1(N) is U_1(20)=0.59909

(ii) When μ_1=4.8, the maximum of U_1(N) is U_1(13)=0.46516

(iii) When μ_1=4.7, the maximum of U_1(N) is U_1(10)=0.37201



we now discuss one more application of the care taker who again conducts series of 'Bernouli trials' to check if the system is in the state N (i.e. number of customers present) upon arrival or otherwise. Thus the success probability is $\pi_0\rho_1^N$= P(system is in the state N) and that of failure event is (1-$\pi_0\rho_1^N$). If this trial is replicated N number of times then the expected value is given by U(N) of (3.15).

The maximum value of U(N) for λ=5.0, $\mu_1 \in \{5.1,5.2,5.3\}$, μ_1=1.1 is represented in the Figure2 where N varies from 1 to 40. From the Figure2b we observe the following pairs (N, U(N)) that produce the expected maximum for these fixed input values λ=5.0, μ_1=1.1 while $\mu_1 \in \{5.1,5.2,5.3\}$:

(i) When μ_1=5.1, the maximum of U(N) is U(20)=0.66778

(ii) When μ_1=5.2, the maximum of U(N) is U(13)=0.58722

(iii) When μ_1=5.3, the maximum of U(N) is U(10)=0.53928

4 Optimal N-policy:Graphical Method

This section considers a real life application of the queueing model M/ M(μ_1)+M(μ_2)/1 under the proposed N-policy. Fixing the revenue per unit time wait, a maximization problem is now formulated as a funtion of N. An optimal value N* for N is determined that maximizes the average revenue per unit time wait.

Considered is a DTP(Desk-Top Publishing) computer center with only one slow speed computer(SSC) and one fast speed computer(FSC). For example, a computer with Windows-10 runs faster than a computer with Windows-XP because Windows-10 supports all the latest hardware and is more optimized.

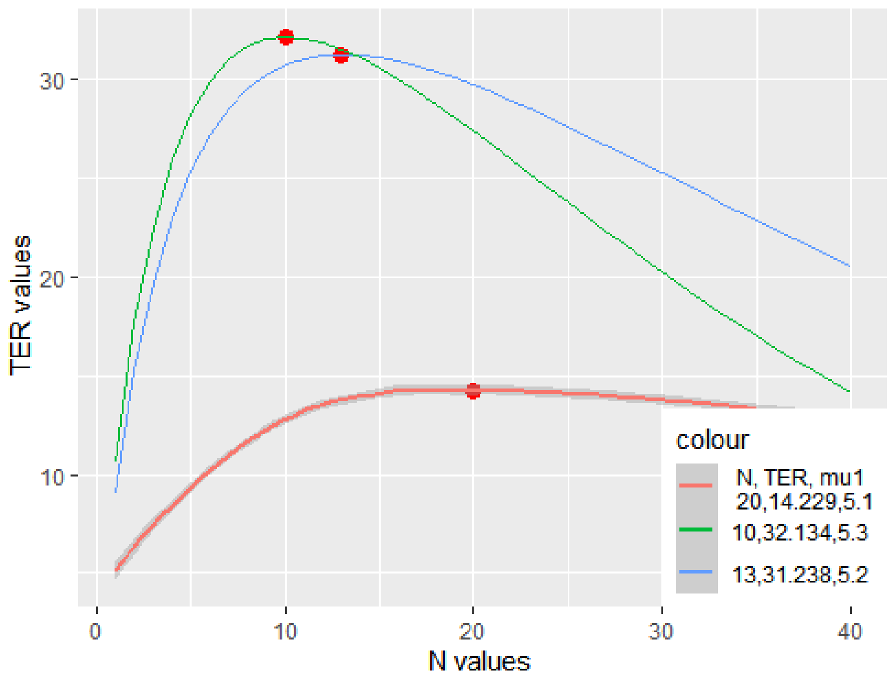
Here the Poisson arrivals or jobs (*i.e. to be converted into a new artwork or text by a computer*) occur with rate λ. The service time needed by SSC is exponentially distributed with a mean time ($1/\mu_1$). On the other hand, the service time required by the FSC exponentially distributed with a mean time ($1/\mu_2$).

The Desktop Publisher(Server) follows an N-policy (as proposed in the preceeding sections): if the queue length remains below or drops from above to the thrshold level N the service completion rate of SSC is μ_1 . If the queue length crosses the level N and reaches (N+1), the FSC attends the tasks with the increased rate $\mu_2 > \mu_1$ in order to reduce the queue length.

Assume that the processing profits are \$ C_1 per unit time wait when the slow sevice rate is μ_1 and it is fixed as \$ $C_2 > C_1$ per unit time wait when the service rate is μ_2. For determining a suitable N-policy that maximizes the total expected revenue (TER) function is

$$TER = TER(N, \lambda, \mu_1, \mu) = U(N)\{\frac{C_2}{\mu_1\,(1-\rho)} - \frac{C_1}{\mu_1\,(1-\rho_1)}\} \tag{4.1}$$

Figure 3: N vs TER values



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For an illustration, we select at random, the following data set:λ= 5.0 , μ=1.1 C_1=8, C_2=100 , μ_1=5.1,5.2 and 5.3. The vector of TER values of (4.1) for each N $\in$ {1 to 45} is computed. The graph representing TER shown in Figure3 ensures that the maximum revenues occur at (N=20, TER=14.229 for μ_1=5.1), (N=23, TER=31.238 for μ_1=5.2), and (N=10, TER=32.134 for μ_1=5.3)

5 Conclusion

Conditional queue length(s) and waiting time(s) for an M/ M(μ_1)+M(μ_2)/1/N-policy model were obtained in the preceding sections. Here we identified an ideal condition on the value of N of the N-policy to transit between a fast rate μ_2 and a slow rate $\mu_1 < \mu_2$ in order to reduce the mean sojourn time.

A link funtion U(N) was proposed to connect the steady state chracteristics of an M/ M(μ_1)/1 queue with another M/M(μ_2)/1 queue controlled by the same N-policy. If the number Q of customers present in the system $\in \Omega_1$={0,1, . . .,N }, customers are served by M/ M(μ_1)/1 or otherwise customers are served by M/ M(μ_2)/1 (i.e. when Q $\in \Omega_2$={(N+1), (N+2), . . ., ∞ }). Customers arrive one by one according to a Poisson process with parameter λ.

For a given N value of the N-policy, we established the Little's formula for the customers of M/ M(μ_1)+ M(μ_2)/1/N-policy queues subject to $\rho_1 = \frac{\lambda}{\mu_1} \neq 1$ and $\rho = \frac{\lambda}{\mu_2} < 1$. The probability of finding the empty system was denoted by π_0. Then the expected number of trials of a Binomial distribution B(N, π_0) was noted by U_1(N)= N π_0. Based on a given input data set on (λ=5, μ_1, μ_2=1.1), U_1(N) was shown as a concave funtion of N together with its maximum value using graphical method.

The probability of finding the system size of M/ M(μ_1)+ M(μ_2)/1 queue at state N was denoted by $\pi_0\ \rho_1^N$. Then the expected number of trials of a Binomial distribution B(N, $\pi_0\ \rho_1^N$) was dnoted by U(N)= U_1(N) ρ_1^N. Based on a given input data set on (λ=5, μ_2=1.1), using another graphical representation between N and U(N), maximum value of U(N) was displayed for some select vaues of μ_1, and N.

A real life aplication is discussed by fixing the revenue per unit time wait, a maximization problem is formulated as a funtion of N. An optimal value N* for N is determined that maximizes the average revenue per unit time wait.

Employing a suitable methodology, time dependent n anlysis can be carried out for the proposed Poisson queue. There is a scope to extend the proposed queueing model to increase the service rate at a state N_1 >N if $\rho_1 = \frac{\lambda}{\mu_1} > 1 \ \ and \ \ \rho = \frac{\lambda}{\mu_2} > 1$ and to the cases of G/G/c/N-policy queues.

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